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Global shift operators and the higher order calculus of variations

David Betounes

Mathematics Department, University of Southern Mississippi, Hattiesburg, MS 39406-5045, USA [1]

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In this paper we prove the existence of global shift operators S associated with any fiber bundle $\pi : E \to M$, and we discuss the use of these operators in the higher order calculus of variations. We use a recent formulation of the variational theory which combines shift operators together with another fundamental operator, called the omega operator, to describe the major aspects of the higher order theory: in particular the Euler operator and the various Cartan operators. This approach provides, we believe, a simple and direct treatment of the subject.

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1. Introduction

While many of the problems connected with formulating the variational theory for higher orders were resolved some years ago, there has been continued interest in simplifying the constructions involved and understanding the basic mechanisms at work. In this vein we recently advocated [Be91] an approach which requires just a minimum number of fundamental objects for formulating the theory: basically just the omega operator Ω and shift operators S. Here in this paper we prove the existence of global shift operators S, which were previously introduced in a purely axiomatic fashion, and we describe additional features of our approach which follow easily from the axioms and which give some of the well-known results in the literature.

In essence the omega operator Ω allows one to describe the Cartan forms and write the variational equations in terms of Cartan forms in a convenient way. In conjunction with Ω, the choice of a shift operator S gives an axiomatic

[1] betounes@usmcp6.bitnet

construction of the Euler operator: $\mathcal{E} = \mathcal{E}_S$ (which does not depend on the S chosen) and the various Cartan operators: $\mathcal{C}_S$ (which in general do depend on S, but only up to a trivial operator). Specifically for a kth order variational problem, these operators are defined in terms of S by

$$\mathcal{E} = (1 + \Omega dS)^k, \tag{1}$$

$$\mathcal{C}_S = (1 + S\Omega d)^k, \tag{2}$$

where d is the exterior derivative. The omega operator is closely related to the horizontalization operator, which has been widely used in the literature, while shift operators, in a certain sense, provide a global mechanism for integration by parts, and by iteration yield the Euler operator as in (1). The constructions in (1), (2) rely on the use of certain vector-valued forms on the various jet spaces: $\pi^k : E^k \equiv J^k E \to M$ of E, which are related to those used by Kolar [Ko84a,Ko84b] and Ferraris and Francaviglia [FF84] (among others) for similar purposes. In our case, we consider the space of contact-horizontal forms: $\mathcal{CH}^n E^{k+1} \subseteq \bigwedge^1 E^{k+1} \otimes \bigwedge^n E^{k+1}$, consisting of the contact one-forms on E^{k+1}, which are π^{k+1}_k-horizontal and which have values in the bundle of π^{k+1}-horizontal n-forms on E^{k+1} (here $n = \dim M$). The diagram in fig. 1 illustrates the overall construction with the domains and codomains of S, Ω, $\mathcal{E}$, and $\mathcal{C}_S$.

Clearly each of the small squares in the diagram is commutative, and thus one has

$$\mathcal{E}\Omega d = \Omega d \mathcal{C}_S. \tag{3}$$

This equation can be taken as the fundamental defining relation for a kth order *Cartan operator* $\mathcal{C}$ in general: $\mathcal{E}\Omega d = \Omega d\mathcal{C}$, with $\mathcal{C} : \bigwedge^n E^k \to \bigwedge^n E^{2k}$.

These constructions naturally express the intrinsic variational aspects of the theory. Thus suppose λ is an n-form on E^k and let $A_\lambda(\sigma) = \int_M \sigma^{k*}\lambda$ be the corresponding action integral, where $\sigma : M \to E$ is a section ($\sigma \in \Gamma E$), and $\sigma^k : M \to E^k$ its k-jet. Then the global variational equations (Euler–Lagrange equations) are

$$\sigma^{2k+1*}\mathcal{E}\Omega d\lambda = 0, \tag{4}$$

or, in terms of the Cartan form $\mathcal{C}_S\lambda$ for λ,

$$\sigma^{2k+1*}\Omega d\mathcal{C}_S\lambda = 0. \tag{5}$$

In certain cases these equations reduce to ones of order less than $2k + 1$, since, depending on the nature of λ, both $\mathcal{E}\Omega d\lambda$ and $\mathcal{C}_S\lambda$ may be pullbacks of forms on lower order jet bundles. (The operators denoted by 1 in the diagram in fig. 1 are shorthand for pullbacks by the various projections: $\pi^{k+1}_k, \ldots, \pi^{2k+1}_{2k}$, and the shift operator vanishes on rank zero forms.)

The above is a brief outline of the approach to the variational theory developed in ref. [Be91] (we provide additional details in section 2). This technique of constructing global Cartan forms via shift operators provides an alternative

$$
\begin{tikzcd}
\Lambda^n E^{2k} \arrow[r, "\Omega d"] & \mathcal{CH}^n E^{2k+1} \\
\Lambda^n E^{2k-1} \arrow[u, "1 + S\Omega d"] \arrow[r, "\Omega d"] & \mathcal{CH}^n E^{2k} \arrow[u, "1 + \Omega d S"'] \arrow[ul, "S"'] \\
\vdots & \vdots \\
\Lambda^n E^{k+2} \arrow[r, "\Omega d"] & \mathcal{CH}^n E^{k+3} \\
\Lambda^n E^{k+1} \arrow[u, "1 + S\Omega d"] \arrow[r, "\Omega d"] & \mathcal{CH}^n E^{k+2} \arrow[u, "1 + \Omega d S"'] \arrow[ul, "S"'] \\
\Lambda^n E^{k} \arrow[u, "1 + S\Omega d"] \arrow[r, "\Omega d"] \arrow[uuuuu, bend left=60, "\mathcal{C}_S"] & \mathcal{CH}^n E^{k+1} \arrow[u, "1 + \Omega d S"'] \arrow[ul, "S"'] \arrow[uuuuu, bend right=60, "\mathcal{E}"']
\end{tikzcd}
$$

Fig. 1. The overall construction with the domains and codomains of S, Ω, $\mathcal{E}$, and $\mathcal{C}_S$.

to other techniques in the literature. These techniques include the use of (1) partitions of unity by Krupka [Kr84,Kr87], (2) connections on the base space by Kolar [Ko 84a,b] and Anderson [A89], (3) pairs of connections by Munoz Masqué [MM85], (4) fibered connections by Ferraris [F84], (5) reduction to first order by Saunders [S89], and (6) ideas from exterior differential systems by Gotay [G91a,G91b]. Some of the earlier work on the resolution of the higher order Cartan form problem includes refs. [FF83,GM83,HK83,Kr83] (see references therein for the prior history of this). We believe our approach here is simpler, more natural and direct, but in any regard, certainly derivative of and founded upon these earlier works in the literature.

In section 3 we discuss shift operators axiomatically and in general, while in section 4 we specialize to what we call basic shift operators. In the latter case it is easy to show that for order $k = 1$, or for $\dim M = 1$, the corresponding Cartan operator is unique: $\mathcal{C}_S = \mathcal{C}_{S'}$, for any two basic shift operators S, S'. In section 5 we show, in general, that a global (basic) shift operator S can be constructed using a torsion-free connection ∇ on M (as well as a volume form ϖ on M). For $\dim M = 1$ only the volume form is required, for $k = 1$ the shift operator S does not depend on ∇, and for $k = 2$ the corresponding Cartan operator $\mathcal{C}_S$ does not depend on ∇.

2. Background and definitions

For the sake of completeness we include here some of the discussion and definitions from ref. [Be91].

Throughout the sequel $n = \dim M$ is the dimension of the base space, which we assume is oriented by a volume form ϖ. On a given coordinate chart we let (ϖ) denote the local function on M given by

$$(\varpi)(x) = \varpi_x(\partial/\partial x_1|_x, \ldots, \partial/\partial x_n|_x).$$

Thus locally $\varpi = (\varpi) dx_1 dx_2 \cdots dx_n$.

2.1. CONTACT HORIZONTAL FORMS

In the sequel it will be convenient to phrase certain constructions in terms of n-form-valued one-forms on jet bundles (cf. refs. [FF84,Ko84a,Ko84b], as mentioned in the introduction). Thus let $\Lambda^1 E^{k+1} \otimes \Lambda^n E^{k+1}$ denote the bundle of one-forms on E^{k+1} with values in the n-form bundle $\Lambda^n E^{k+1}$, and let $CH^n E^{k+1} \subseteq \Lambda^1 E^{k+1} \otimes \Lambda^n E^{k+1}$ be the subbundle with elements (u, ϕ_u), $u \in E^{k+1}$ and ϕ_u satisfying:

$$\phi_u(X_u)(Z_u^1, \ldots, Z_u^n) = 0, \tag{6}$$

whenever (i) one of the Z_u^i's is π^{k+1}-vertical ($d\pi^{k+1}|_u Z_u^i = 0$) or (ii) X_u is π_k^{k+1} vertical, or (iii) $X_u = d\sigma^{k+1}|_x Y_x$ for some section σ with $\sigma^{k+1}(x) = u$ and $Y_x \in T_x M$. Thus a section $\phi : E^{k+1} \to CH^n E^{k+1}$ of this subbundle, $\phi \in \mathcal{CH}^n E^{k+1}$, is just a contact one-form with values in the horizontal n-forms and which also vanishes, $\phi_u(X_u) = 0$, on π_k^{k+1}-vertical vectors X_u. In local coordinates ϕ has the expression:

$$\phi = \sum_{|\alpha|=0}^{k} \phi_\alpha^a \omega_\alpha^a \otimes \Delta, \tag{7}$$

where $\omega_\alpha^a \equiv dy_\alpha^a - y_{\alpha+e_j}^a dx_j$ are the basic contact one-forms on $E^{k+1}, \Delta = dx_1 dx_2 \cdots dx_n$, and $\phi_\alpha^a \equiv \phi(\partial/\partial y_\alpha^a)(\partial/\partial x_1, \ldots, \partial/\partial x_n)$. Our notation here (and throughout the sequel) is as follows: $\{x_i\}_{i=1}^n, \{y_\alpha^a\}_{|\alpha|=0,\ldots,k+1}^{a=1,\ldots,m}$ denote coordinate functions on a fibered chart of E^{k+1}, with m = the dimension of the fiber of E. Also $\alpha = (\alpha_1, \ldots, \alpha_n) \in N_0^n$ is a multi-index, $|\alpha| = \alpha_1 + \cdots + \alpha_n$, and $e_j = (0, \ldots, 1, \ldots, 0)$ is the multi-index with jth entry 1 and zeros elsewhere. We also use implied summation on repeated indices, with the one exception that summation over a range of orders of multi-indices, like $\sum_{|\alpha|=0}^k$ in the above expression, is explicitly indicated.

The rank $\mathrm{rk}(\phi)$ of a contact horizontal form ϕ is the least of all the integers $r \in \{0, 1, \ldots, k\}$ for which $\phi_u(X_u) = 0$, $\forall u$ and $\forall X_u \in V_u\pi_r^{k+1}$ (the π_r^{k+1}-vertical vectors). Thus if ϕ has rank r, then locally $\phi = \sum_{|\alpha|=0}^r \phi_\alpha^a \omega_\alpha^a \otimes \Delta$.

2.2. THE OMEGA OPERATOR

The omega operator is the operator $\Omega : \wedge^{n+1}E^k \to \mathcal{CH}^n E^{k+1}$ (one for each $k = 1, 2, \ldots$), defined by

$$\begin{aligned}\Omega(\Psi)_u(X_u)&(Z_u^1, \cdots, Z_u^n) \\ &= \Psi_z(d\pi_k^{k+1}|_u X_u, d\sigma^k|_x\, d\pi^{k+1}|_u Z_u^1, \ldots, d\sigma^k|_x\, d\pi^{k+1}|_u Z_u^n), \qquad (8)\end{aligned}$$

where $u = [\sigma]_x^{k+1}$ and $z = \pi_k^{k+1}(u)$. Some of the properties of Ω are: (1) If X' is a vector field on E^{k+1} which projects to X on E^k then $\Omega(\Psi)(X') = \mathrm{hor}(X \lrcorner \Psi)$, where hor is the horizontalization operator (cf., e.g., ref. [Kr83], p. 199) and $\lrcorner$ denotes contraction (the interior operator). (2) Locally on a jet chart $\Omega(\Psi)$ is given by

$$\Omega(\Psi) = \sum_{|\alpha|=0}^{k} \Omega_\alpha^a(\Psi)\omega_\alpha^a \otimes \Delta,$$

where

$$\begin{aligned}\Omega_\alpha^a(\Psi) &= \Omega(\Psi)(\partial/\partial y_\alpha^a)(\partial/\partial x_1, \ldots, \partial/\partial x_n) \\ &= \mathrm{hor}(\partial/\partial y_\alpha^a \lrcorner \Psi)(\partial/\partial x_1, \ldots, \partial/\partial x_n)\end{aligned}$$

are the components.

The omega operator demarcates the set of *Cartan forms* on E^k: these are the n-forms C on E^k such that ΩdC has rank zero. (In the literature these forms are usually called Lepagean forms, as suggested by Krupka [Kr73], due to Lepage's work [Le36].) In particular, an n-form τ on E^k such that $\Omega d\tau = 0$ is called a *trivial form*. If $\lambda \in \wedge^n E^k$ is an n-form, then a *Cartan form for* λ is a Cartan form C such that $\Omega dC = \mathcal{E}\Omega d\lambda$. Thus the set of Cartan forms for λ on E^k is $\{C_0 + \tau | \tau \in \ker \Omega d\}$ where C_0 is any particular Cartan form for λ. For a Cartan form C on E^k, the variational equations (Euler–Lagrange equations) are simply $\sigma^{k+1*}\Omega dC = 0$.

Generally the variational problem of interest begins with a Lagrangian form $\lambda = L\varpi$ on E^k, determined by an appropriate Lagrangian $L : E^k \to R$. In this case the local expression for $\Omega d\lambda$ is:

$$\Omega d\lambda = \sum_{|\alpha|=0}^{k} (\varpi)\frac{\partial L}{\partial y^a_\alpha}\omega^a_\alpha \otimes \Delta, \tag{9}$$

and the variational equations are locally given by:

$$\sum_{|\alpha|=0}^{k} (-1)^{|\alpha|}\left(\frac{\partial}{\partial x}\right)^{\alpha}\left[(\varpi)\frac{\partial L}{\partial y^a_\alpha} \circ \sigma^k\right] = 0 \quad (a = 1, \ldots, m). \tag{10}$$

This is expressed globally by using either the Euler operator: $\sigma^{2k+1*}\mathcal{E}\Omega d\lambda = 0$, or a Cartan operator: $\sigma^{2k+1*}\Omega d\mathcal{C}\lambda = 0$ (next section). The nature of the pullbacks for these vector-valued forms is explained in ref. [Be91].

3. Shift operators in general

To define the Euler operator and various Cartan operators we use the notion of shift operators.

Definition. A shift operator is a mapping

$$S : \mathcal{CH}^n E^{k+1} \to \Lambda^n E^{k+1}$$

(one for each $k = 1, 2, \ldots$), with the following property: for each $\phi \in \mathcal{CH}^n E^{k+1}$

(1) if $\operatorname{rk}\phi = 0$, then $S\phi = 0$;

(2) if $\operatorname{rk}\phi \geq 1$, then $\operatorname{rk}(\phi + \Omega dS\phi) \leq \operatorname{rk}\phi - 1$.

In (2) we identify, notationally, ϕ with its pullback $\pi^{k+2*}_{k+1}\phi$, and we will generally write $\phi + \Omega dS\phi = (1 + \Omega dS)\phi$. The associated Euler and Cartan operators are defined by $\mathcal{E}_S\phi = (1 + \Omega dS)^k\phi$ and $\mathcal{C}_S\lambda = (1 + S\Omega d)^k\lambda$. It has been shown [Be91] that the Euler operator does not depend on S (that is, $\mathcal{E}_S = \mathcal{E}_{S'}$, for any two shift operators S, S'), and the the local expression for $\mathcal{E}$ is

$$\mathcal{E}\phi = \left(\sum_{|\alpha|=0}^{k} (-1)^{|\alpha|} D^\alpha \phi^a_\alpha\right)\omega^a \otimes \Delta. \tag{11}$$

Here $D^\alpha = D_1^{\alpha_1} D_2^{\alpha_2} \cdots D_n^{\alpha_n}$, where D_i is a local differential operator

$$D_i = \frac{\partial}{\partial x_i} + \sum_{|\alpha|=0}^{2k} y^a_{\alpha+e_i}\frac{\partial}{\partial y^a_\alpha}.$$

The existence of an operator $\mathcal{E}$ with property (11) does not really rely on the existence of global shift operators (local shift opertors suffice for the existence proof, or other methods from the literature can be used).

The Cartan operator $\mathcal{C}_S\lambda = (1+S\Omega d)^k\lambda$ is more easily computed by iteration: let

$$\begin{aligned} \phi_0 &= \phi = \Omega d\lambda, \\ \phi_1 &= (1+\Omega dS)\phi, \\ \phi_2 &= (1+\Omega dS)\phi_1, \\ &\vdots \\ \phi_{k-1} &= (1+\Omega dS)\phi_{k-2}. \end{aligned}$$

Then

$$\mathcal{C}_S\lambda = \lambda + S\phi + S\phi_1 + \cdots + S\phi_{k-1}. \tag{12}$$

It should be noted that, depending on the rank of $\phi = \Omega d\lambda$, the Euler and Cartan forms: $\mathcal{E}\Omega d\lambda$ and $\mathcal{C}_S\lambda$ may reside on lower order jet bundles (i.e., be pullbacks of forms on lower order jet bundles). Thus if $\phi = \Omega d\lambda$ has rank r, then $\phi_r = (1+\Omega dS)^r\phi$ has rank 0 and so $\phi_{r+s} = (1+\Omega dS)^s\phi_r = \phi_r$ (more precisely $= \pi^{k+1+r+s*}_{k+1+r}\phi_r$), for $s = 1,2,...,k-r$. Thus $\mathcal{E}\Omega d\lambda$ and $\mathcal{C}_S\lambda = \lambda + S\phi + \cdots + S\phi_{r-1}$ are pullbacks of forms on E^{k+1+r} and E^{k+r}. In addition when $\lambda = L\varpi$ is a Lagrangian form it is easy to see from (11) that $\mathcal{E}\Omega d\lambda$ resides on E^{2k} (at most), and for the special types of shift operators discussed in the next section one has that $\mathcal{C}_S\lambda$ resides on E^{2k-1} (at most).

4. Basic shift operators

In this section we restrict attention to a special class of shift operators, called basic shift operators, and their corresponding basic Cartan operators. Before doing this, however, we make the following observations about the general case.

(1) The set of shift operators is convex.

(2) Any two Cartan operators differ by a trivial operator, i.e., if $\mathcal{C}_1, \mathcal{C}_2$ satisfy $\Omega d\mathcal{C}_1 = \mathcal{E}\Omega d = \Omega d\mathcal{C}_2$, then $\mathcal{C}_1 - \mathcal{C}_2$ is a trivial operator (has values in $\ker\Omega d$). In particular, for any two shift operators S, S':

$$\mathcal{C}_S = \mathcal{C}_{S'} + \mathcal{T},$$

where $\mathcal{T}$ is a trivial operator.

(3) If S is a shift operator and T is a trivial operator ($T : \mathcal{CH}^nE^{k+1} \to \ker\Omega d$, $k = 1,2,...$), then $S+T$ is a shift operator, and $\mathcal{C}_{S+T}\lambda = \mathcal{C}_S(\lambda) + T(\phi) + \cdots + T(\phi_{k-1})$. Here $\phi = \Omega d\lambda$ and $\phi_r = (1+\Omega dS)^r\phi$.

(4) If C is any Cartan form for $\lambda \in \wedge^nE^k$, then $C = \mathcal{C}_S\lambda$ for some shift operator S. To see this let S' be any shift operator. Then $\tau \equiv C - \mathcal{C}_{S'}\lambda \in \wedge^nE^{2k}$ is a trivial form, and defining $T(\phi) = \tau$ for $\phi \in \mathcal{CH}^nE^{2k}$ and $T(\phi) = 0$ otherwise, gives a trivial operator. Then $\mathcal{C}_{S'+T}(\lambda) = \mathcal{C}_{S'}\lambda + \tau = C$.

Definition. A shift operator is called *basic* if for every $\phi \in \mathcal{CH}^n E^{k+1}$:

(a) $S\phi$ is one-contact (i.e., $S\phi$ is a contact n-form on E^{k+1} such that $W \lrcorner Z \lrcorner S\phi = 0$ for any two π^{k+1}-vertical vector fields W, Z).

(b) $S\phi$ is $\pi^{k+1}_{\mathrm{rk}\,\phi-1}$-horizontal.

On each jet chart, a basic shift operator S has the local expression, for $\phi \in \mathcal{CH}^n E^{k+1}$:

$$S\phi = \sum_{|\alpha|=0}^{k-1} S^a_{\alpha i}(\phi)\omega^a_\alpha \Delta_i, \tag{13}$$

where

$$\Delta_i = \partial/\partial x_i \lrcorner \Delta = (-1)^{i+1} dx_1 \cdots \widehat{dx_i} \cdots dx_n,$$

and the components of S are given by

$$S^a_{\alpha i}(\phi) = (-1)^{i+1}(S\phi)(\partial/\partial y^a_\alpha, D_1, ..., \widehat{D_i}, ..., D_n). \tag{14}$$

The expression (13) is the general local expression for S, but for any particular ϕ, condition (b) gives that the summation in (13) need only extend to $|\alpha| = \mathrm{rk}\,\phi - 1$, since $S^a_{\alpha i}(\phi) = 0$, for $|\alpha| = \mathrm{rk}\,\phi, ..., k-1$. The Cartan operator $\mathcal{C}_S$ for a basic shift operator has the local expression:

$$\mathcal{C}_S\lambda = \lambda + \sum_{|\alpha|=0}^{k-1}\left(\sum_{r=0}^{k-1-|\alpha|} S^a_{\alpha i}(\phi_r)\right)\omega^a_\alpha \Delta_i, \tag{15}$$

where as usual we put: $\phi_r = (1 + \Omega dS)^r \phi$, with $\phi = \Omega d\lambda$.

For basic shift operators the condition that $\phi + \Omega dS\phi$ have rank at least one less than that of ϕ is analyzed locally as follows.

First for $\mathrm{rk}\,\phi = 1$, one has $S\phi = S^a_{0i}(\phi)\omega^a\Delta_i$, and

$$\phi + \Omega dS\phi = [\phi^a - D_i S^a_{0i}(\phi)]\,\omega^a \otimes \Delta + [\phi^a_i - S^a_{0i}(\phi)]\,\omega^a_i \otimes \Delta. \tag{16}$$

For this to be rank 0, one must have: $S^a_{0i}(\phi) = \phi^a_i$, which shows that the action on rank 1 forms, and thus on $\mathcal{CH}^n E^1$, is the same for all basic shift operators. In addition, for $\lambda \in \bigwedge^n E^1$, the Cartan form is $\mathcal{C}_S\lambda = \lambda + S\phi = \lambda + \phi^a_i\omega^a\Delta_i$.

Next for $\mathrm{rk}\,\phi \geq 2$, a short computation gives the local expression:

$$\begin{aligned}\phi + \Omega dS\phi = {} & (\phi^a - D_i S^a_{0i}(\phi))\,\omega^a \otimes \Delta \\ & + \sum_{|\alpha|=1}^{k-1}\left(\phi^a_\alpha - \sum_{\mu+e_i=\alpha} S^a_{\mu i}(\phi) - D_i S^a_{\alpha i}(\phi)\right)\omega^a_\alpha \otimes \Delta \\ & + \sum_{|\alpha|=k}\left(\phi^a_\alpha - \sum_{\mu+e_i=\alpha} S^a_{\mu i}(\phi)\right)\omega^a_\alpha \otimes \Delta.\end{aligned} \tag{17}$$

In the formula, the summations $\sum_{\mu+e_i=\alpha}$ stand for the summation over all pairs (μ, i) such that $\mu + e_i = \alpha$. Also note that, since $S^a_{\alpha i}(\phi) = 0$ whenever $|\alpha| \geq \mathrm{rk}\,\phi$, one can reduce the outer summations in formula (17) to ones with k replaced by $\operatorname{rk}\phi$. Thus to satisfy the rank condition on shift operators one must have locally:

$$\sum_{\mu+e_i=\alpha} S^a_{\mu i}(\phi) = \phi^a_\alpha, \tag{18}$$

for each α with $|\alpha| = \operatorname{rk}\phi$. There are a number of natural candidates for *local* shift operators which satisfy condition (18), but the problem is to obtain a global one satisfying the condition. In the next section we give a relatively straightforward construction of a global basic shift operator S such that on each jet chart:

$$S^a_{\mu i}(\phi) = \frac{\mu_i + 1}{\operatorname{rk}\phi}\,\phi^a_{\mu+e_i}, \tag{19}$$

for $|\mu| = \operatorname{rk}\phi - 1$ (and $\operatorname{rk}\phi \geq 1$).

For the case of $\operatorname{rk}\phi = 2$, condition (18) says that any basic shift operator satisfies:

$$S^a_{e_j i}(\phi) + S^a_{e_i j}(\phi) = \phi^a_{e_i+e_j} \quad (i \neq j), \tag{20}$$

$$S_{e_j j}(\phi) = \phi^a_{2e_j}. \tag{21}$$

From this, and the action of S on rank ones, it is easy to see that for $\lambda \in \Lambda^n E^2$, the expression for $\mathcal{C}_S\lambda = \lambda + S\phi + S\phi_1$ is locally given by:

$$\mathcal{C}_S\lambda = \lambda + \left(\phi^a_i - D_j S^a_{e_i j}(\phi)\right)\omega^a\Delta_i + S^a_{e_j i}(\phi)\,\omega^a_j\Delta_i. \tag{22}$$

For the shift operator we construct in the next section, which uses a connection ∇ on M, one sees from (19) that

$$S^a_{e_j i}(\phi) = \frac{\delta_{ij} + 1}{\operatorname{rk}\phi}\,\phi^a_{e_i+e_j},$$

when $1 \leq \operatorname{rk}\phi \leq 2$. Thus the corresponding second order basic Cartan operator has local expression:

$$\mathcal{C}_S\lambda = \lambda + \left(\phi^a_i - \frac{\delta_{ij} + 1}{\operatorname{rk}\phi}\,D_j\phi^a_{e_j+e_i}\right)\omega^a\Delta_i \tag{23}$$

$$+ \frac{\delta_{ij} + 1}{\operatorname{rk}\phi}\,\phi^a_{e_j+e_i}\,\omega^a_j\Delta_i.$$

Thus $\mathcal{C}_S\lambda$ does not depend on the connection ∇ used in defining S. This result together with the assertions in the following theorem have occurred often in the literature in various forms and have been derived by different methods (cf., e.g., refs. [F84,GM83,G91a,Ko84a,Kr83,MM85,S89]). By our method, the results in theorem 1 are derived by just using *local* shift operators.

Theorem 1.

(1) *Suppose $\lambda \in \wedge^n E^{k+1}$ is any n-form and $C \in \wedge^n E^{2k}$ is a Cartan form for λ such that $C - \lambda$ is one-contact and π^{2k}_{k-1}-horizontal (i.e. C is a* basic *Cartan form for λ). Then on each chart the local expression for C is:*

$$C = \lambda + \sum_{|\alpha|=0}^{k-1} \left(\sum_{|\beta|=0}^{k-1-|\alpha|} M^{\beta}_{\alpha i} D^{\beta} \phi^a_{\alpha+\beta+e_i} + \tau^a_{\alpha i} \right) \omega^a_{\alpha} \Delta_i. \tag{24}$$

Here $\phi = \Omega d\lambda$, and the constants $M^{\beta}_{\alpha i}$ are:

$$M^{\beta}_{\alpha i} = (-1)^{|\beta|} \frac{|\alpha|!|\beta|!(\alpha + \beta + e_i)!}{\alpha!\beta!(|\alpha| + |\beta| + 1)!}, \tag{25}$$

while the $\tau^a_{\alpha i}$'s are the components of a trivial form on E^{2k}.

(2) *For base dimension $n = 1$, or for order $k = 1$, any trivial form on E^{2k}, which is one-contact and π^{2k}_{k-1}-horizontal, is identically zero. Consequently in either of these cases there is at most one basic Cartan form C for each λ, and in addition $C_S = C_{S'}$ for any two basic shift operators S, S'. For $n = 1$ the expression* (24) *reduces to:*

$$C = \lambda + \sum_{\alpha=0}^{k-1} \left(\sum_{\beta=0}^{k-1-\alpha} (-1)^{\beta} D^{\beta} \phi^a_{\alpha+\beta+1} \right) \omega^a_{\alpha}. \tag{26}$$

Proof.

(1) To get the local expression (24) for C on a chart, define a *local* basic shift operator $\tilde{S}$ on this chart by:

$$\tilde{S}\phi = \sum_{|\alpha|=0}^{k-1} \frac{(\alpha + e_i)!}{\alpha!(|\alpha| + 1)} \phi^a_{\alpha+e_i} \omega^a_{\alpha} \Delta_i. \tag{27}$$

The computation of $C_{\tilde{S}}\lambda$ is particularly easy since $\tilde{S}$ has the property that $\sum_{\mu+e_i=\alpha} \tilde{S}^a_{\mu i}(\phi) = \phi^a_{\alpha}$, for every $\alpha \neq 0$. Thus formula (17) simplifies to:

$$\begin{aligned}
\phi_1 &= \phi + \Omega d \tilde{S} \phi \\
&= (\phi^a - D_i \tilde{S}^a_{0i}(\phi))\, \omega^a \otimes \Delta - \sum_{|\alpha|=1}^{k-1} D_i \tilde{S}^a_{\alpha i}(\phi)\, \omega^a_{\alpha} \otimes \Delta \\
&= (\phi^a - D_i \phi^a_i)\, \omega^a \otimes \Delta - \sum_{|\alpha|=1}^{k-1} \frac{(\alpha + e_i)!}{\alpha!(|\alpha| + 1)} D_i \phi^a_{\alpha+e_i}\, \omega^a_{\alpha} \otimes \Delta.
\end{aligned} \tag{28}$$

From this one obtains:

$$\tilde{S}\phi_1 = - \sum_{|\alpha|=0}^{k-2} \frac{(\alpha + e_i + e_j)!}{\alpha!(|\alpha| + 1)(|\alpha| + 2)} D_j \phi_{\alpha+e_i+e_j}\, \omega^a_{\alpha} \Delta_i. \tag{29}$$

Continuing with this iterative calculation: $\phi_2 = \phi_1 + \Omega d\tilde{S}\phi_1$, $\tilde{S}\phi_2$, ... , one arrives at the *locally defined* Cartan form constructed from $\tilde{S}$:

$$\mathcal{C}_{\tilde{S}}\lambda = \lambda + \sum_{|\alpha|=0}^{k-1} \left(\sum_{|\beta|=0}^{k-1-|\alpha|} M_{\alpha i}^{\beta} D^{\beta} \phi_{\alpha+\beta+e_i}^{a} \right) \omega_{\alpha}^{a} \Delta_i. \tag{30}$$

Now since any two Cartan forms for λ (local or global) differ by a trivial form, one has in particular:

$$C = \mathcal{C}_{\tilde{S}}\lambda + \tau, \tag{31}$$

where τ is a local trivial form on E^{2k}. Since both C and $\mathcal{C}_{\tilde{S}}\lambda$ are basic, τ necessarily has the expression: $\tau = \sum_{|\alpha|=0}^{k-1} \tau_{\alpha i}^{a} \omega_{\alpha}^{a} \Delta_i$. Thus formula (31) is the same as (24).

(2) Suppose $\tau \in \bigwedge^n E^{2k}$ is one-contact and π_{k-1}^{2k}-horizontal. Locally $\tau = \sum_{|\alpha|=0}^{k-1} \tau_{\alpha i}^{a} \omega_{\alpha}^{a} \Delta_i$, and is trivial ($\Omega d\tau = 0$) if and only if

$$\begin{aligned} D_i \tau_{0i}^{a} &= 0, \\ D_i \tau_{\alpha i}^{a} + \sum_{\mu+e_i=\alpha} \tau_{\mu i}^{a} &= 0 \quad (0 < |\alpha| < k), \\ \sum_{\mu+e_i=\alpha} \tau_{\mu i}^{a} &= 0 \quad (|\alpha| = k). \end{aligned} \tag{32}$$

Thus for $n = 1$, or for $k = 1$, any such trivial form is identically zero. But then from formula (24), one sees that any two basic Cartan forms C, C' for λ coincide locally with $\mathcal{C}_{\tilde{S}}\lambda$, and thus $C = C'$. □

Definition. The local shift operator $\tilde{S}$ defined by formula (27) in the above proof is called the *canonical local shift operator*. This operator is purely local (does not globalize in general), but is useful in computing the local expression (24) for any basic Cartan form for λ.

There are other candidates for local shift operators for which $\mathcal{C}_S\lambda$ is easily computed, for instance:

$$\bar{S}\phi = \sum_{|\alpha|=0}^{k-1} \frac{1}{N(\alpha + e_i)} \phi_{\alpha+e_i}^{a} \omega_{\alpha}^{a} \Delta_i, \tag{33}$$

where $N(\beta)$ denotes the number of non-zero components of β. However, the use of $\tilde{S}$ gives formula (24), which coincides with the formula given by Munoz Masqué [MM85] [when specialized to the Lagrangian case: $\lambda = L\varpi$, so that $\phi_{\alpha}^{a} = (\varpi)\partial L/\partial y_{\alpha}^{a}$]. Other versions of Cartan forms from the literature can be obtained in a similiar fashion (the differences lying in the trivial form τ). One anomaly occurs here. For the global shift operator S we construct in the next section, the local computation of $\mathcal{C}_S\lambda$ is exceedingly difficult (even for $n = 1$), and so it is perhaps best to be satisfied with knowing:

$$\mathcal{C}_S\lambda = \mathcal{C}_{\widetilde{S}}\lambda + \tau, \tag{34}$$

in terms of the canonical local shift operator $\widetilde{S}$ and some local trivial form τ [which vanishes for $n = 1$ (and $k = 1$)].

5. Existence of global shift operators

In this section we prove the existence of global shift operators for any fiber bundle $\pi : E \to M$, by constructing a specific one which is basic and which satisfies condition (19) on each jet chart. The construction involves an operator $B(\phi, g)$ which is similar to the Saunders operator S_ω (cf. ref. [S89], p. 235). We do not use his operator here, but rather proceed along a slightly different route.

Thus let γ be the canonical contact structure on E^{k+1}, i.e., for a vector field Z on E^{k+1}, $\gamma(Z)$ is the π^{k+1}-vertical vector field along π_k^{k+1} defined by (cf. ref. [S89], p. 215):

$$\gamma(Z)_u = d\pi_k^{k+1}|_u Z_u - d\sigma^k|_x d\pi^{k+1}|_u Z_u, \tag{35}$$

where $u = [\sigma]_x^{k+1}$ and $x = \pi^{k+1}(u)$. Thus locally one has:

$$\gamma(Z)_u = \sum_{|\alpha|=0}^{k} \omega_\alpha^a(Z)_u \ \partial/\partial y_\alpha^a|_z, \tag{36}$$

where $z = \pi_k^{k+1}(u)$.

As before, let ϖ be a volume form on M, and identify ϖ with is pullback $\pi^{k+1*}\varpi$. For an n-form $\theta_u \in \wedge^n E_u^{k+1}$, let

$$*\theta_u = \frac{\theta_u(\partial/\partial x_1|_u, ..., \partial/\partial x_n|_u)}{\varpi_u(\partial/\partial x_1|_u, ..., \partial/\partial x_n|_u)}.$$

Here $x_1, ..., x_n$ are coordinate functions on M (lifted to E^{k+1}), and the definition does not depend on the choice of coordinates.

Theorem 2. *Suppose $\phi \in \mathcal{CH}^n E^{k+1}$ and $g \in C^\infty M$. Let $B(\phi, g)$ denote the contact one-form on E^{k+1} defined by: for $u \in E^{k+1}$ and $Z_u \in T_u E^{k+1}$*

$$B(\phi, g)_u(Z_u) = *\phi_u\left(\left(\tilde{g}Y\right)_u^{k+1}\right), \tag{37}$$

where Y is a vertical vector field on E such that

$$\gamma(Z)_u = Y_u^{k+1} \tag{38}$$

(here Y^{k+1} denotes the prolongation of Y to E^{k+1}), and $\tilde{g} = g - g(x)$, with $x = \pi^{k+1}(u)$ (we identify g with its pullback to E).

The definition of $B(\phi, g)$ in (37) does not depend on the Y chosen to satisfy (38). Furthermore on any jet chart one has the local expression:

$$B(\phi, g) = \sum_{|\alpha|=0}^{k-1} B_\alpha^a(\phi, g)\omega_\alpha^a, \tag{39}$$

where

$$B_\alpha^a(\phi, g) = \sum_{|\beta|=0}^{k-1-|\alpha|} \frac{\binom{\alpha+\beta+e_i}{\alpha}}{N(\beta+e_i)\ (\varpi)} [(\partial/\partial x)^{\beta+e_i} g]\phi_{\alpha+\beta+e_i}. \tag{40}$$

Here $\binom{\alpha}{\nu} = \alpha!/[\nu!(\alpha-\nu)!]$ and $N(\alpha)$ = the number of non-zero components of $\alpha = (\alpha_1, \ldots, \alpha_n)$. Note also that in (40) there is implied summation on $i = 1, \ldots, n$, and in (39) and (40) one can replace k by $\operatorname{rk}\phi$.

Proof. First, it is clear that for a given point $u \in E^{k+1}$, there is a vertical vector field Y on E such that $Y_u^{k+1} = \gamma(Z)_u$. To see this, note that on a given chart about u, the requirement is: $D^\alpha Y^a = \omega_\alpha^a(Z)_u$, for $a = 1, \ldots, m$ (where $D^\alpha = D_1^{\alpha_1} D_2^{\alpha_2} \cdots D_n^{\alpha_n}$). On the chart one can construct such a Y which is constant on the fibers [so that $D^\alpha Y^a = (\partial/\partial x)^\alpha Y^a$], and then extend this to all of E.

Next, in order to see that the definition (37) does not depend on the Y so chosen, look at the local expression for $(\tilde{g}Y)^{k+1}$ on a chart about u, and use Leibnitz's rule to get:

$$\begin{aligned}
(\tilde{g}Y)_u^{k+1} &= \left[\sum_{|\mu|=0}^{k+1} D^\mu(\tilde{g}Y^a)\frac{\partial}{\partial y_\mu^a}\right]_u \\
&= \left[\sum_{|\alpha|=0}^{k+1}\sum_{|\nu|=0}^{k+1-|\alpha|}\binom{\alpha+\nu}{\alpha} D^\nu\tilde{g}\, D^\alpha Y^a \frac{\partial}{\partial y_{\alpha+\nu}^a}\right]_u \\
&= \sum_{|\alpha|=0}^{k}\left(\sum_{|\beta|=0}^{k-|\alpha|}\frac{\binom{\alpha+\beta+e_i}{\alpha}}{N(\beta+e_i)}\left(\frac{\partial}{\partial x}\right)^{\beta+e_i} g\Big|_x\right)\omega_\alpha^a(Z)_u \frac{\partial}{\partial y_{\alpha+\beta+e_i}}\Big|_u
\end{aligned} \tag{41}$$

The above expression (41) shows that $(\tilde{g}Y)_u^{k+1}$ does not depend on the choice of Y. Furthermore by evaluating $\phi_u((\tilde{g}Y)_u^{k+1})$, using expression (41) and the fact that $\phi_{\alpha+\beta+e_i} = 0$, for $|\alpha| + |\beta| = k$, one arrives at the local coordinate expression (39), (40) for $B(\phi, g)$. □

Remark. The Saunders operator S_ω (cf. ref. [S89], p. 235) is essentially given by the expression in eq. (41).

The theorem shows that $B(\phi, g)_u(Z_u)$ only depends on the k-jet $g^k(x)$ of g at $x = \pi^{k+1}(u)$ (where g is considered as a section $g : M \to M \times R$.) To construct a shift operator out of this, it is natural to look at something of the form:

$$B(\phi, g^{Z_u^1 \cdots Z_u^{n-1}})_u(Z_u^n), \tag{42}$$

where $(Z_u^1, \ldots, Z_u^{n-1}) \mapsto g^{Z_u^1 \cdots Z_u^{n-1}} \in C^\infty M$ is a multilinear selection of a function on M. It is not clear whether there is a natural multilinear selection to use, or whether such a selection yields a smooth tensor field on E^{k+1}, defined via eq. (42). In addition to achieve a shift operator out of such a construction the $|\alpha| = \operatorname{rk}\phi - 1$ components of $B(\phi, g)$ should have a specific form. The following theorem shows one way to construct $S\phi$ out of $B(\phi, g)$, somewhat along these lines.

Theorem 3. *There exists a global basic shift operator, constructed as follows: Let $n = \dim M$.*

(1) For $n = 1$, define $S\phi$ by:

$$(S\phi)_u(Z_u) = \frac{1}{\operatorname{rk}\phi} B(\phi, g)_u(Z_u), \tag{43}$$

where $u \in E^{k+1}$, $Z_u \in T_u E^{k+1}$, and g is a function on M, which on a neighborhood of $x = \pi^{k+1}(u)$ satisfies:

$$dg = \varpi. \tag{44}$$

The definition does not depend on the choice of g, and the components of $S\phi$ are smooth functions on E^{k+1}:

$$S_\alpha^a(\phi) = \frac{\alpha+1}{\operatorname{rk}\phi}\phi_{\alpha+1}^a + \sum_{\beta=1}^{k-1-\alpha} \frac{\binom{\alpha+\beta+1}{\alpha}}{\operatorname{rk}\phi} C_{\beta+1}(\varpi)\phi_{\alpha+\beta+1}^a, \tag{45}$$

for $\alpha = 0, 1, \ldots, k-1$, and by convention the summation $\sum_{\beta=1}^{k-1-\alpha}$ is absent when $\alpha = k-1$. In formula (45)

$$C_{\beta+1}(\varpi) \equiv \frac{1}{(\varpi)}(d/dx)^\beta(\varpi) \qquad (\varpi)(x) \equiv \varpi_x(d/dx|_x).$$

(2) For $n \geq 2$, define $S\phi$ as follows. Let ∇ be a torsion-free connection on M. For a multilinear form θ on a vector space, let $\#\theta$ denote the symmetrization of θ. For $u \in E^{k+1}$ and $Z_u^1, \ldots, Z_u^n \in T_u E^{k+1}$, choose a function

$$g = g^{Z_u^1 \cdots Z_u^{n-1}}$$

on M which satisfies:

$$dg^{Z_u^1 \cdots Z_u^{n-1}}|_x = (-1)^{n-1}(\tilde{Z}_x^{n-1} \lrcorner \cdots \lrcorner \tilde{Z}_x^1 \lrcorner \varpi_x), \tag{46}$$

$$\#\nabla^p dg^{Z_u^1 \cdots Z_u^{n-1}}|_x = 0, \tag{47}$$

for $p = 1, ..., k$. *Here* $\widetilde{Z}_x^j = d\pi^{k+1}|_u Z_u^j$ *and* $\nabla^p = \nabla \circ \nabla \circ \cdots \circ \nabla$ *denotes the application of the covariant derivative p times. Then*

$$B(\phi, g^{Z_u^1 \cdots Z_u^{n-1}})_u (Z_u^n) \tag{48}$$

does not depend on the choice of g satisfying eqs.(46), (47), and the antisymmetrization of this expression gives a basic shift operator:

$$(S\phi)_u (Z_u^1, ..., Z_u^n) = \frac{1}{n!} \sum_{\pi} \frac{(-1)^{\pi}}{\operatorname{rk} \phi} B(\phi, g^{Z_u^{\pi 1} \cdots Z_u^{\pi(n-1)}})(Z_u^{\pi n}), \tag{49}$$

with local expression on each jet chart given by $S\phi = \sum_{|\alpha|=0}^{k-1} S_{\alpha j}^a (\phi) \omega_\alpha^a \Delta_j$. *The components of this basic shift operator are:*

$$S_{\alpha j}^a (\phi) = \frac{\alpha_j + 1}{\operatorname{rk} \phi} \phi_{\alpha + e_j}^a + \sum_{|\beta|=0}^{k-1-|\alpha|} \frac{\binom{\alpha + \beta + e_i}{\alpha}}{\operatorname{rk} \phi \, N(\beta + e_i)} C_{\beta + e_i}^j (\varpi, \nabla) \phi_{\alpha+\beta+e_i}^a, \tag{50}$$

for $|\alpha| = 0, ..., k-1$, *and by convention the summation* $\sum_{\beta=1}^{k-1-|\alpha|}$ *is absent when* $|\alpha| = k-1$. *These components are smooth functions on* E^{k+1} *with* $C_{\beta+e_i}^j (\varpi, \nabla)$ *depending on* ϖ *and the derivatives of the connection components in general.*

Proof.

(1) This part follows from theorem 2.

(2) First note that for a given $u \in E^{k+1}$, and $Z_u^1, ..., Z_u^{n-1}$, eqs. (46), (47) constitute an intrinsic way of specifying the k-jet of $\tilde{g} = g - g(x)$ at x. Thus any two functions $g, \bar{g}$ which satisfy these equations give the same B operator: $B(\phi, g)_u = B(\phi, \bar{g})_u$ at u. Further, the multilinear dependence on $Z_u^1, ..., Z_u^{n-1}$ of any solution is clear from the form of the equations . Thus it suffices to look at these equations locally and exhibit a local solution. Equation (46) is

$$\partial g / \partial x_i |_x = (-1)^{n-1} \varpi_x (\widetilde{Z}_x^1, ..., \widetilde{Z}_x^{n-1}, \partial / \partial x_i |_x) \tag{51}$$

and eqs. (47) for $p = 1, ..., k$, recursively specify the higher order derivatives of g at x in terms of $\partial g(x) / \partial x_r$ and the derivatives of the connection components. Namely, by looking at

$$(\nabla d g)_{ij} = \frac{\partial^2 g}{\partial x_i \partial x_j} - \Gamma_{ij}^r \frac{\partial g}{\partial x_r}, \tag{52}$$

$$(\nabla^2 d g)_{ijc} = \frac{\partial}{\partial x_c} \left(\frac{\partial^2 g}{\partial x_i \partial x_j} - \Gamma_{ij}^r \frac{\partial g}{\partial x_r} \right)$$
$$-\Gamma_{ic}^s \left(\frac{\partial^2 g}{\partial x_j \partial x_s} - \Gamma_{js}^r \frac{\partial g}{\partial x_r} \right) - \Gamma_{jc}^s \left(\frac{\partial^2 g}{\partial x_i \partial x_s} - \Gamma_{is}^r \frac{\partial g}{\partial x_r} \right), \tag{53}$$

etc., one sees that g satisfies eqs. (47) if and only if

$$\left.\frac{\partial^s}{\partial x_{c_1}\cdots\partial x_{c_s}}\frac{\partial^2 g}{\partial x_i\,\partial x_j}\right|_x = \#\left.\frac{\partial^s}{\partial x_{c_1}\cdots\partial x_{c_s}}\left(\Gamma^r_{ij}\frac{\partial g}{\partial x_r}\right)\right|_x \tag{54}$$

$$= P^r(\Gamma)\,\partial g/\partial x_r|_x, \tag{55}$$

for each $s = 1, 2, ..., k-1$. Here the $\#$ indicates symmetrization of the indices $c_1, ..., c_s, i, j$, and $P^r(\Gamma)$ is a certain polynomial depending on Γ and its derivatives up to order s. This polynomial results from eq. (54) by applying Leibnitz' rule there and recursion to eliminate all but the first order derivatives of g.

All of this shows that the expression

$$B(\phi, g^{Z^1_u\cdots Z^{n-1}_u})_u(Z^n_u) \tag{56}$$

does not depend on the choice of g, and is multilinear in $Z^1_u, ..., Z^n_u$ (with actual dependence only on the projections of these to T_xM).

The antisymmetrization of (56) (divided by $\operatorname{rk}\phi$), gives an n-form $S\phi$ on E^{k+1} which is clearly one-contact and π^{k+1}_k-horizontal. To see that it is a shift operator we compute $S^a_{\mu j}(\phi)$ for $|\mu| = \operatorname{rk}\phi - 1$:

$$\begin{aligned}
S^a_{\mu j}(\phi)_u &= (-1)^{j+1}(S\phi)_u(\partial/\partial y^a_\mu|_u, D_1(u)\cdots\widehat{D_j(u)}\cdots D_u(u)) \\
&= (-1)^{j+1}B(\phi, g^{D_1(u)\cdots\widehat{D}_j(u)\cdots D_n(u)})_u(\partial/\partial y^a_\mu|_u)/\operatorname{rk}\phi \\
&= (-1)^{j+1}\sum_{i=1}^n \frac{\mu_i+1}{(\varpi)(x)}\frac{\partial}{\partial x_i}g^{D_1(u)\cdots\widehat{D}_j(u)\cdots D_n(u)}|_x\phi^a_{\mu+e_i}(u)/\operatorname{rk}\phi \\
&= \frac{(-1)^{n-1}\cancel{(-1)^{j+1}}}{\operatorname{rk}\phi\,(\varpi)(x)}\sum_{i=1}^n(\mu_i+1) \\
&\quad\times\varpi_x(\partial/\partial x_1|_x, ..., \widehat{\partial/\partial x_j}|_x, ..., \partial/\partial x_n|_x, \partial/\partial x_i|_x)\phi^a_{\mu+e_i}(u) \\
&= \frac{\mu_j+1}{\operatorname{rk}\phi}\phi^a_{\mu+e_j}(u).
\end{aligned} \tag{57}$$

This calculation (57) uses formula (40) with $\alpha = \mu$ and $\beta = 0$ (since $|\mu| = \operatorname{rk}\phi - 1$). Similarly one can use formula (40) to calculate the other components of S and arrive at formula (50). □

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